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Technical Note

Analytical solutions of Nusselt number for thermally fully developed flow in microtubes under a combined action of electroosmotic forces and imposed pressure gradients

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Abstract

Closed form expressions are developed for Nusselt number variation in a thermally fully developed microtube flow, under a combined influence of electroosmotic forces and imposed pressure gradients. The analysis takes care of the interaction amongst pressure driven convection and Joule heating effects, in order to obtain the pertinent rate of heat transfer. While separate limiting conditions on the asymptotic Nusselt number can be obtained for pure electroosmotic and solely pressure driven flows, relative influences of electrical potential gradients and imposed pressure gradients acting in tandem are also critically analyzed, as a function of the tube radius normalized with respect to the Debye length. Significant insights are also developed regarding the influence of adverse pressure gradients on the thermal transport, in presence of aiding electroosmotic effects.

Fluid flow in channels of micron or submicron length scales has received serious attention from research community in the recent past, primarily attributable to pathbreaking developments in the areas of microelectronics and MEMS, microfluidics-based biomedical separation and diagnostic techniques, leading to various 'lab-on-a-chip' applications. In this respect, electroosmosis [1–6] has been extensively used as a driving force to manipulate liquid flows and to transport and control liquid samples of nanovolumes in microfluidic devices, used for biochemical and biotechnological applications as well as thermal management of microelectronics devices. However, so far, most of the pertinent investigations have been primarily focused on fluid flow behaviour, and not a great attention has been paid on influence of flow behaviour on the thermal transport.

Off late, it has been recognized that high electrical fields employed for electroosmotic transport create both conduction and convection currents in the liquid. The convection current contributes to a net flow in the system, whereas the conduction current generates a volumetric Joule heating. Such Joule heating effects may not only cause local enhancements in temperature values, but also can create high temperature gradients. Previous studies have demonstrated that that these effects can result in low column separation efficiency, reduction of analysis resolution, and even loss of injected samples in biomedical applications. In addition, a temperature rise can lead to decomposition of thermally labile samples and formation of gas bubbles. In microelectronic devices, such effects may also result in inefficient heat dissipation, leading to local overheating problems. Irrespective of such significant consequences, thermal energy generation and the associated dissipation mechanisms have received only very little attention in the context of electroosmotic and pressure-driven microflows, as apparent from reported research investigations.

In classical macro-flow domains, explicit analytical solutions have been reported in the past for developing as well as fully developed thermal transport problems with or

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without volumetric heat generation. In a pioneering work, Sparrow et al. [7] have studied the effect of an arbitrary generation term in pressure driven flows, and have obtained analytical solutions for the pertinent heat transfer characteristics. Houriuchi and Dutta [8], as well as Maynes and Web [9], in their recent independent studies, have reported analytical solutions of Nusselt number for steady electroosmotic flows in two dimensional microchannels. However, their studies have been restricted to pure electroosmotically driven flows only. In reality, on the other hand, many microflows are combined electroosmotic and pressure driven in nature. In the literature, however, no study has been reported so far, providing close formed expressions for Nusselt number in such complicated situations.

Aim of the present work, accordingly, is to develop analytical expressions for Nusselt number in a thermally fully developed microtube flow, under a combined influence of electroosmotic forces and imposed pressure gradients. The analysis takes care of the interaction amongst pressure driven convection and Joule heating effects, in order to obtain the pertinent rate of heat transfer. This is expected to provide significant theoretical insights in designing micropumps, valves and mixers, by appealing to close formed expressions depicting interrelationships between various significant parameters that dictate overall heat transfer rates through an effective Nusselt number.

In presence of an applied pressure gradient and an applied electric potential, equation for fully developed fluid motion is as:

$$\frac{\mathrm{d}^2 v_z}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v_z}{\mathrm{d}r} = -\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} + \frac{1}{\mu}\rho_e E_z \tag{1}$$

where v_z is the axial velocity component, p is the pressure, μ is the viscosity, and E_z is the axial potential gradient. In Eq. (1), ρ_e is distribution of excess charge density, which is obtained from a simultaneous solution of the Poisson's equation of potential distribution, in conjunction with the Boltzmann equation of charge density distribution. For low wall potential, the Debye–Hückel linearization is valid [10]. In such circumstances, the wall zeta potential (ζ) may assumed to be a constant and less than $3k_\mathrm{B}T$ (where k_B is the Bolzmann constant), and ρ_e can be expressed explicitly as a function of only r, ζ, and λ (where λ is the Debye length). This leads to the following solution of Eq. (1):

$$v_z(r) = \frac{-1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\left(R^2 - r^2\right) + u_\mathrm{HS}\left(1 - \frac{I_0\left(\frac{r}{\lambda}\right)}{I_0\left(\frac{R}{\lambda}\right)}\right) \tag{2}$$

where R is the radius of the microtube. In Eq. (2), the term u_HS represents the maximum possible electroosmotic velocity (Helmholtz–Smoluchowski electroosmotic velocity) for a given applied potential field, and is described as:

$$u_\mathrm{HS} = -\frac{\varepsilon\zeta}{\mu}E_z \tag{2a}$$

where ε is the permittivity of the medium.

Based on the above solution of velocity field, the thermal energy equation can be used to obtain the temperature distribution within the fluid. Under a simplified assumption of constant thermophysical properties, the above equation takes the following form:

$$\rho c_p v_z \frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + k\frac{\partial^2 T}{\partial z^2} + \sigma E_z^2 + \mu\phi \tag{3}$$

where ρ is the density, c_p is the specific heat at constant pressure, k is the thermal conductivity, σ is the electrical conductivity and ϕ is the viscous dissipation function (which is proportional to square of the velocity gradient). In Eq. (3), the third term in RHS represents a volumetric heat generation due to electric resistance heating (Joule heating) and the next term represents a local volumetric heating due to viscous dissipation. For an assessment of relative order of magnitude of these two terms, one may obtain a ratio of strength of Joule heating and viscous dissipation as: $R_\mathrm{v} \sim \frac{\sigma R \mu \lambda}{\varepsilon^2\zeta^2}$. Typically, $\lambda \sim 10^{-8}$ m, $\sigma \sim 10^{-3}$ S/m, $\mu \sim 10^{-3}$ N s/m^2, $\varepsilon \sim 10^{-9}$ C/V m, $\zeta \sim -100$ mV, which imply that $R_\mathrm{v} \sim 10^6 R$. Hence, it can be concluded that the Joule heating dominates relative to viscous dissipation if R is O(10 μm) or higher. For the present study, we assume R to be larger than 10 μm, and the viscous dissipation is, therefore, negligible in comparison to Joule heating effects.

Further simplifications in Eq. (3) can be made by imposing specific boundary conditions at the tube wall. Here, we consider the boundary condition of a constant wall heat flux (q''_w), which together with a thermally fully developed flow condition [11] yields:

$$\frac{\partial T}{\partial z} = \frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z} = \frac{\partial T_\mathrm{w}}{\partial z} = \text{constant} \tag{3a}$$

where T_m is the bulk mean temperature of flow and T_w is the tube wall temperature distribution. Under these conditions, Eq. (3) can be integrated twice to yield:

$$T = \frac{\rho c_p \lambda^2}{k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\left[\begin{array}{c} -\frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\left(\frac{R^2\chi^2}{4} - \frac{\lambda^2\chi^4}{16} - \frac{3R^2a^2}{16}\right) \\ +u_\mathrm{HS}\left\{\frac{\chi^2 - a^2}{4} - \frac{I_0(\chi)}{I_0(a)} + 1\right\} \end{array}\right] + \frac{E_z^2\lambda^2\sigma(a^2 - \chi^2)}{4k} + T_\mathrm{w} \tag{4}$$

where $\chi = \frac{r}{\lambda}$ and $a = \frac{R}{\lambda}$. Next, we appeal to the definition of T_m as:

$$T_\mathrm{m} = \frac{\int_0^R v_z T r\,\mathrm{d}r}{\int_0^R v_z r\,\mathrm{d}r} \tag{5}$$

and utilize the wall boundary condition as:

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=R} = h(T_\mathrm{m} - T_\mathrm{w}) \tag{6}$$

h being the convective heat transfer coefficient, to obtain:

$$Nu_{2R}=\frac{h(2R)}{k}=\frac{2R\left[-\rho c_p\lambda\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\left\{-\frac{R^2a}{16\mu}\frac{\mathrm{d}p}{\mathrm{d}z}+u_\mathrm{HS}\left(\frac{a}{2}-\frac{I_1(a)}{I_0(a)}\right)\right\}+\frac{E_z^2R\sigma}{2}\right]\left(\frac{-R^4}{16\mu}\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{Au_\mathrm{HS}R^2}{a^2}\right)}{kB} \tag{7}$$

where

$$A=\frac{a^2}{2}-a\frac{I_1(a)}{I_0(a)} \tag{7a}$$

and

$$\begin{aligned}B={}&\frac{-11R^4a^4\lambda^4\rho c_p}{6144\mu^2k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2+\frac{R^2a^4\lambda^4u_\mathrm{HS}\rho c_p}{96\mu k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\frac{\mathrm{d}p}{\mathrm{d}z}\\&+\frac{R^2\lambda^4u_\mathrm{HS}\rho c_p}{4\mu k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\frac{\mathrm{d}p}{\mathrm{d}z}\left[2-2a\frac{I_1(a)}{I_0(a)}-\frac{a^2}{4}\right]\\&-\frac{E_z^2\lambda^4R^2a^4\sigma}{96\mu k}\frac{\mathrm{d}p}{\mathrm{d}z}-\frac{\lambda^4R^2\rho c_p}{4\mu k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\frac{\mathrm{d}p}{\mathrm{d}z}\\&\times\left[\frac{a^2}{4}-\frac{a^4}{24}-2+\left(\frac{a}{2}+\frac{4}{a}\right)\frac{I_1(a)}{I_0(a)}\right]\\&+\left[\frac{R^2\lambda^2u_\mathrm{HS}^2\rho c_p}{4k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}-\frac{E_z^2R^2\lambda^2\sigma u_\mathrm{HS}}{4k}\right]\left[2-2a\frac{I_1(a)}{I_0(a)}-\frac{a^2}{4}\right]\\&+\frac{\lambda^4u_\mathrm{HS}^2\rho c_p}{k}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\left[a^2-\frac{a^2}{2}\left\{\frac{I_1(a)}{I_0(a)}\right\}^2-2a\left\{\frac{I_1(a)}{I_0(a)}\right\}\right]\end{aligned} \tag{7b}$$

In Eq. (7), the term $\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}$ can be calculated by executing an overall energy balance for an elemental control volume to obtain:

$$\dot{m}c_p\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}=2\pi q''_\mathrm{w}R+\pi R^2\sigma E_z^2 \tag{8}$$

where $\dot{m}=\int_0^R\rho(2\pi rv_z\mathrm{d}r)$. On simplification, we get

$$\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}=\frac{q''_\mathrm{w}+\sigma E_z^2R/2}{\rho c_p\left[-\frac{R^3}{8\mu}\frac{\mathrm{d}p}{\mathrm{d}z}+u_\mathrm{HS}\left\{\frac{R}{2}-\lambda\frac{I_1(a)}{I_0(a)}\right\}\right]} \tag{9}$$

Eq. (7), in conjunction with Eq. (9), establishes a closed form expression for Nusselt number, in terms of significant physical parameters governing the overall thermofluidic transport. Based on these derivations, we can now analyze a number of limiting special cases, some of which are as follows:

(i) For a pure pressure driven flow (i.e., $u_\mathrm{HS}=0$, $E_z=0$), we get, from Eq. (7), $Nu_{2R}=48/11$, which is a well-known classical result for convective heat transfer in a pressure-driven thermally fully developed flow in a circular tube with a constant wall heat flux.

(ii) For a pure electroosmotic flow ($\mathrm{d}p/\mathrm{d}z=0$), Eq. (7) yields:

$$Nu_{2R}=\frac{2u_\mathrm{HS}\left[\frac{a}{2}-\frac{I_1(a)}{I_0(a)}\right]\left[\frac{E_z^2\sigma}{2}-\frac{\rho c_pu_\mathrm{HS}}{a}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\left\{\frac{a}{2}-\frac{I_1(a)}{I_0(a)}\right\}\right]}{\left[\frac{\rho c_pQu_\mathrm{HS}^2}{4a}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}+\frac{\rho c_pu_\mathrm{HS}^2}{a^3}\frac{\mathrm{d}T_\mathrm{m}}{\mathrm{d}z}\left\{a^2-\frac{a^2}{2}\left(\frac{I_1(a)}{I_0(a)}\right)^2-2a\frac{I_1(a)}{I_0(a)}\right\}-\frac{\sigma u_\mathrm{HS}E_z^2Q}{4a}\right]} \tag{10}$$

where $Q=2-2a\frac{I_1(a)}{I_0(a)}-\frac{a^2}{4}$. The above leads to the classical theoretical result of $Nu_{2R}=8$, as $a\to\infty$.

(iii) Under a condition of no applied E_z field, there is a steady state attained when the streaming current, I_s, is balanced by the conduction current, I_c, produced in the reverse direction, i.e.,

$$I_\mathrm{s}+I_\mathrm{c}=0 \tag{11}$$

where $I_\mathrm{s}=\int_0^R\rho_ev_z2\pi r\,\mathrm{d}r$, and $I_\mathrm{c}=\sigma E_z\pi R^2$. From the above expression, we get, for this case,

$$E_z=\frac{-\frac{\varepsilon\zeta}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\left[1-\frac{2I_1(a)}{aI_0(a)}\right]}{\sigma-\frac{\varepsilon^2\zeta^2}{\mu\lambda^2}\left[1-\frac{2I_1(a)}{aI_0(a)}-\left\{\frac{I_1(a)}{I_0(a)}\right\}^2\right]} \tag{12}$$

In order to obtain deeper insights regarding variation of the Nusselt number, the same is first plotted against the ratio a, for different values of the parameter $P=\frac{-R^2\mathrm{d}p/\mathrm{d}z}{4\mu u_\mathrm{HS}}$ (which is a measure of the relative significance of pressure gradient and electroosmotic forces), as depicted in Fig. 1. For all plots in Fig. 1, the parameter $S=\frac{E_z^2\sigma R}{2q''_\mathrm{w}}$, which is a measure of the rate of heat generation due to Joule heating relative to the rate of heat transfer at the tube wall, is kept unaltered at $S=0.01$. It can be observed from Fig. 1 that at $P=0$, the solution tends to the classical limit of 8 as $a\to\infty$. At lower values of P, Nu_{2R} monotonically increases with increase a, and tends to attain a saturation as $a\to\infty$. However, for higher values of P, the above saturation seems to be initiated at lower values of a, and also settles to lower values of asymptotic Nu_{2R}. For $P=10$, however, an interesting feature is observed in a sense that

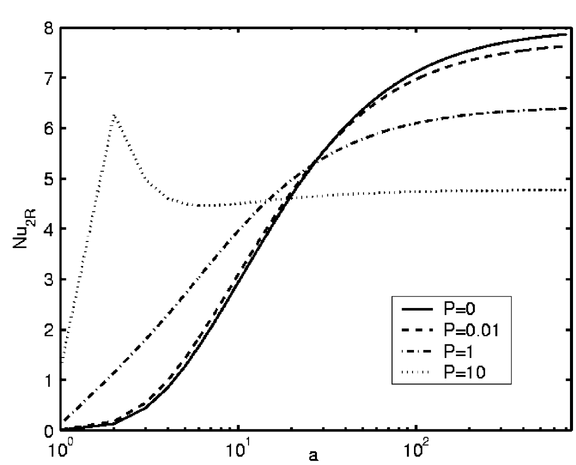

Fig. 1. Variation of Nu_{2R} as a function of a, for different values of P. For all curves, value of S is taken as 0.01.

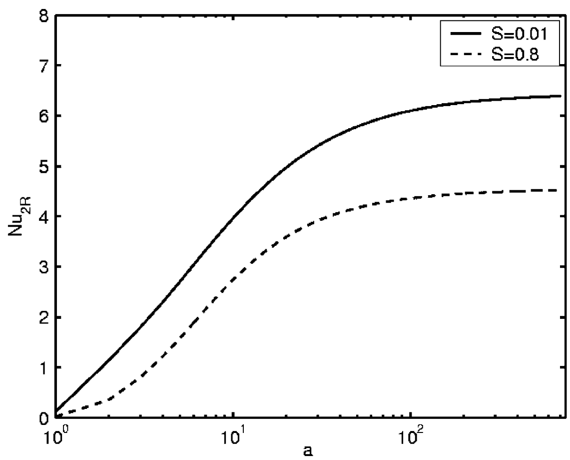

Fig. 2. Variation of Nu_{2R} as a function of a, for different values of S. For all curves, value of P is taken as 1.

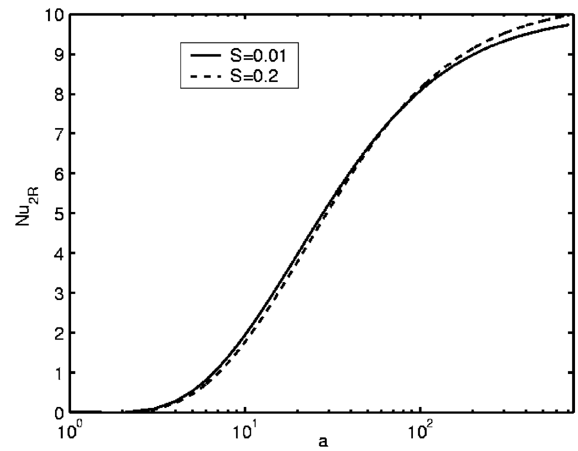

Fig. 3. Variation of Nu_{2R} as a function of a, for different values of S, in presence of an adverse pressure gradient ($P = -0.5$).

there occurs a local peak in the Nu_{2R} value for approximately $a = 2$, beyond which Nu_{2R} suddenly decreases, and eventually settles to a stable value for higher values of a. This can be attributed to the fact that for higher values of P, effect of wall potential distribution on flow characteristics can be strongly felt only when the tube radius is of comparable magnitude relative to the electric double layer (EDL) thickness, leading to a consequent enhancement in overall rate of convective transport. However, as the ratio of tube radius to EDL thickness becomes larger, local charge density gradients within the EDL turn out to be rather inconsequential in determining the Nu_{2R}, which eventually settles to a stable value of ≈ 4.36 as $a \to \infty$.

Next, Nusselt number is plotted as a function of the ratio a, for different values of the parameter S, as depicted in Fig. 2. For these plots, value of the parameter P is kept fixed at unity. It can be observed form Fig. 2 that higher values of S imply lower values of asymptotic Nu_{2R} as $a \to \infty$. This can be attributed to the fact that as S increases, temperature difference between the tube wall and the bulk fluid also increases. For a constant wall heat flux, this implies a reduced value of the convective heat transfer coefficient, and a consequent decrement in value of the asymptotic Nu_{2R}. Moreover, for lower values of the parameter a, a very high rate of volumetric heat generation would eventually mean that temperature rise inside the fluid domain is significantly large (because of the very small fluid volume) in comparison to the wall temperature. This, in turn, implies that the convective heat transfer coefficient tends to zero, to ensure a finite rate of wall heat flux. For lower values of heat generation rate, however, such excessive overheating may not occur, and non-zero values of Nu_{2R} can be obtained even for small values of a, in the limiting sense as $a \to 1$.

Finally, in Fig. 3, we depict an interesting situation when the flow takes place in presence of an adverse pressure gradient ($P = -0.5$), under the influence of driving electroosmotic effects. It can be seen from Fig. 3 that the Nu_{2R} is virtually zero for low values of the ratio a. This can be attributed to the fact that for a fixed value of the parameter P, a very small value of the ratio a (or equivalently, the tube radius, R) implies that $\mathrm{d}p/\mathrm{d}z$ is a sufficiently large negative quantity to arrest any convective transport, leading to very small values of Nu_{2R}. However, for higher values of a, the adverse effects of a negative $\mathrm{d}p/\mathrm{d}z$ turn out to be less prominent, and the Nu_{2R} increases as a consequence.

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